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ENERGY OF PARTIAL COMPLEMENTS OF A GRAPH

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Abstract. The partial complement of a graph G with respect to a set S denoted by $G \oplus S$ is the graph obtained by removing the edges of $\langle S \rangle$ and adding edges which are not in $\langle S \rangle$ in G. In this paper we introduce the concept of energy of partial complements of graph and partial complement energy is computed for few classes of graphs. Some bounds are obtained for partial complement energy of a graph G.

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1. Introduction

One of the fundamental question in graph theory concerns efficiency of recognition of a graph class Ģ. For instance how quick we can predict whether a graph is planar, 2-connected, chordal, triangle free, bipartite, 3-colorable etc. An atomic operation is the change of adjacency required for modifying an input graph into the desired graph class Ģ is one of the main focus of partial complementation. Many problems in graph theory come under this graph modification category. One may be interested to find wheteher it is possible to add or delete atmost k edges to make a graph 2-edge connected or cordal or traingle free etc. In literature, atomic operation is the change of adjacency. i.e., for a pair of vertices x and y, either one can add or delete an edge xy.

The concept of graph energy originates from chemistry to estimate the total π-electron energy of a molecule. In chemistry the conjugated hydrocarbons can be represented by a graph called molecular graph. Here every carbon atom is represented by a vertex and every carbon-carbon bond by an edge and hydrogen atoms are ignored. The eigenvalues of molecular graph represent energy level of electron in molecule. An interesting quantity in Huckel theory is sum of energies of all the electrons in a molecule, so called $\pi-$electron energy of a molecule.

Let G be a graph with n vertices $\{v_1, v_2, \ldots, v_n\}$ and m edges. Let $A = (a_{ij})$ be an adjacency matrix of graph. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A, assumed in non increasing order, are called eigenvalues of G. As A is real symmetric matrix, eigenvalues of G are real with sum equal to zero. The energy of G [5] is defined to be sum of absolute values of the eigenvalues of G. i.e., $E(G) = \sum\limits_{i=1}^{n} |\lambda_i|$. For all terminologies we refer [3, 4, 9].

In 1998 E. Sampathkumar and L. Pushpalatha defined generalised complements of a graph [16, 17]. Recently Fedor V. Fomin and others introduced partial complements of graph [8]. This motivated us to study the energy of partial complement of a graph. Let $G = (V, E)$ be a graph and $S \subseteq V$. The partial complement of a graph G with respect to S, denoted by $G \oplus S$, is a graph (V, E_S), where for any two vertices $u, v \in V$, $uv \in E_S$ if and only if one of the following conditions hold good:

(1) $u \notin S$ or $v \notin S$ and $uv \in E$.
(2) $u, v \in S$ and $uv \notin E$.

Alternatively, we can also define partial complement of graph G with respect to a set S as graph obtained from G by removing edges of $\langle S \rangle$ and adding the edges which are not in $\langle S \rangle$.
Let $G \oplus S$ be partial complement of a graph G with respect to S. We define partial complement adjacency matrix of $G \oplus S$ as $n \times n$ matrix defined by $A_p(G \oplus S) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent with } i \neq j \\ 1, & \text{if } i = j \text{ and } v_i \in S \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

We observe that an adjacency matrix and partial complement adjacency matrix of a graph G are related as follows:
If $A(G) = \left[\begin{array}{c|c} L & M \\ \hline M' & N \end{array}\right]$ then $A_p(G \oplus S) = \left[\begin{array}{c|c} \overline{L} + I & M \\ \hline M' & N \end{array}\right]$ where L is an adjacency matrix of induced subgraph $\langle S \rangle$ and I is an identity matrix.
The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $A_p(G \oplus S)$ is called spectrum of $G \oplus S$. Partial complement energy of $G \oplus S$, denoted by $E_p(G \oplus S)$ is defined as $\sum_{i=1}^{n} |\lambda_i|$. For more information on energy of graph we refer [1, 2, 6, 7, 10, 14, 15, 18, 19]. For convenience, throughout this paper $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $G \oplus S$. The paper is organised as follows. In section 3, the properties of energy of partial complements of a graph are achieved. In section 4, bounds for energy of partial complement of a graph are established. In section 5, energy of partial complements of some families of graphs is computed.

2. A chemical connection

The equation (1) by which partial complement matrix is defined, can be viewed also as definition of an ordinary adjacency matrix of a partial complement of a graph with loops. Indeed, $A_p(G \oplus S)$ is adjacency matrix of a partial complement of G by attaching a loop of weight $+1$ to each of its vertices belonging to the induced set S.

Graphs with loops are natural representations of heteroconjugated molecules, and have been much studied in chemical graph theory. Loops of weight $+1$ are just the graph representation of nitrogen atoms.

The HMO theory of graphs with loops (i. e., molecular graphs of heteroconjugated molecules) were also studied in detail, including the total $\pi-$electron energy in [11, 12, 13].

3. Properties of energy of partial complements of a graph

Theorem 3.1. *Let $G \oplus S = (V, E_S)$ be the partial complement of a graph $G = (V, E)$. Let $\phi\{A_p(G \oplus S), \lambda\} = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + a_3\lambda^{n-3} + \ldots + a_n$ be the characteristic polynomial of graph $G \oplus S$. Then,*

(1) $a_0 = 1$.
(2) $a_1 = -|S|$.
(3) $a_2 = \binom{|S|}{2} - |E_S|$.
(4) $a_3 = |S||E_S| - \binom{|S|}{3} - \sum_{v \in \langle S \rangle} deg\ v - 2\triangle$, *where $\triangle$ is number of triangles in the graph $G \oplus S$.*

Theorem 3.2. *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ represent eigenvalues of $G \oplus S$, then*

(1) $\sum_{i=1}^{n} \lambda_i = |S|$
(2) $\sum_{i=1}^{n} \lambda_i^2 = 2m_S + |S|$, *where m_S is number of edges of $G \oplus S$.*

Theorem 3.3. *Let $G \oplus S_1$ and $H \oplus S_2$ be two partial complements of graph G and H respectively on n vertices. Let m_{S1}, m_{S2} denote the number of edges of $G \oplus S_1$ and $H \oplus S_1$ respectively. If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are eigenvalues of $G \oplus S_1$ and $\lambda_1' \geq \lambda_2' \geq \cdots \geq \lambda_n'$ are eigenvalues of $H \oplus S_1$. Then, $\sum_{i=1}^{n} \lambda_i \lambda_i' \leq \sqrt{(2m_{S1} + |S_1|)(2m_{S2} + |S_2|)}$.*

Theorem 3.4. *Let $G \oplus S$ be the partial complement of a graph G with induced subgraph $\langle S \rangle$. If the partial complement energy $E_p(G \oplus S)$ is rational number, then $E_p(G \oplus S) \equiv |S| (\mod 2)$.*

4. Bounds for Energy of partial complement of a graph

Theorem 4.1. *Let $G \oplus S$ be the partial complement of a graph G on n vertices with induced subgraph $\langle S \rangle$. Then*

(1) $\sqrt{2m_S + |S|} \leq E_p(G \oplus S) \leq \sqrt{n(2m_S + |S|)}$.
(2) $E_p(G \oplus S) \geq \sqrt{2m_S + |S| + n(n-1)D^{2/n}}$ *where $D = |A_p(G \oplus S)|$.*

Theorem 4.2. *Let $G \oplus S = (V, E_S)$ be a connected graph of order n and size m_S with induced subgraph $\langle S \rangle$. Then*

$$\sqrt{4m_S + |S|(2 - |S|)} \leq E_p(G \oplus S) \leq \sqrt{2m_S(2m_S + |S|)}.$$

Theorem 4.3. *Let $\rho(G \oplus S)$ be the spectral radius of $G \oplus S$ of order n and size m_S. Then*

$$\sqrt{\frac{2m_S + |S|}{n}} \leq \rho(G \oplus S) \leq \sqrt{2m_S + |S|}.$$

Proof. Consider

$$\begin{aligned}\rho^2(G \oplus S) &= \max_{1\le i\le n}\{|\lambda_i|^2\}\\ &\le \sum_{j=1}^{n}\lambda_j^2 = 2m_S + |S|.\\ \rho(G\oplus S) &\le \sqrt{2m_S + |S|}.\end{aligned}$$

Next consider

$$\begin{aligned}n\rho^2(G \oplus S) &\ge \max_{1\le i\le n}\{|\lambda_i|^2\}\\ &\ge 2m_S + |S|.\end{aligned}$$

Thus,

$$\rho(G\oplus S) \ge \sqrt{\frac{2m_S+|S|}{n}}.$$

Hence, $\sqrt{\frac{2m_S+|S|}{n}} \le \rho(G\oplus S) \le \sqrt{2m_S+|S|}$. □

Theorem 4.4. *If $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n$ are the eigenvalues of $G \oplus S$ on n vertices and m_S edges, then $E_p(G\oplus S) \le \lambda_1 + \sqrt{(n-1)(2m_S+|S|-\lambda_1^2)}$.*

Proof. Applying Cauchy Schwarz inequality for $(n-1)$ terms,

$$\begin{aligned}\left(\sum_{i=2}^{n}\lambda_i\right)^2 &\le \left(\sum_{i=2}^{n}1\right)\left(\sum_{i=2}^{n}\lambda_i^2\right)\\ [E_p(G\oplus S)-\lambda_1]^2 &\le (n-1)(2m_S+|S|-\lambda_1^2)\\ E_p(G\oplus S) &\le \lambda_1 + \sqrt{(n-1)(2m_S+|S|-\lambda_1^2)}.\end{aligned}$$

□

Theorem 4.5. *For $G\oplus S$ on n vertices, m_S edges and $2m_S \ge n$,*

$$E_p(G\oplus S) \le \frac{2m_S+|S|}{n} + \sqrt{(n-1)\left[2m_S+|S|-\left(\frac{2m_S+|S|}{n}\right)^2\right]}.$$

Proof. From Theorem 4.4, we have,

$$E_p(G\oplus S) \le \lambda_1 + \sqrt{(n-1)(2m_S+|S|-\lambda_1^2)}.$$

Let

$$f(x) = x + \sqrt{(n-1)(2m_S+|S|-x^2)}.$$

For decreasing function

$$\begin{aligned}f'(x) \le 0 &\Rightarrow 1 - \frac{2x(n-1)}{2\sqrt{(n-1)(2m_S+|S|-x^2)}} \le 0\\ &\Rightarrow x \ge \sqrt{\frac{2m_S+|S|}{n}}.\end{aligned}$$

Since $2m_S + |S| \geq n$, we have, $\sqrt{\frac{2m_S+|S|}{n}} \leq \frac{2m_S+|S|}{n} \leq \lambda_1$

Thus, $E_p(G \oplus S) \leq \frac{2m_S+|S|}{n} + \sqrt{(n-1)\left[2m_S+|S| - \left(\frac{2m_S+|S|}{n}\right)^2\right]}$. □

Lemma 4.6. [15] *Let $a, a_1, a_2, \ldots a_n, A$ and $b, b_1, b_2, \ldots b_n, B$ be real numbers such that $a \leq a_i \leq A$ and $b \leq b_i \leq B$ $\forall i = 1, 2, \ldots n$. Then the following inequality is valid. .*

$$\left| n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \right| \leq \alpha(n)(A-a)(B-b) \tag{2}$$

and equality holds if and only if $a_1 = a_2 = \ldots = a_n$ and $b_1 = b_2 = \ldots = b_n$.

Theorem 4.7. *Let $|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|$ be non-increasing order of eigenvalues of $A_p(G \oplus S)$. Then $E_p(G \oplus S) \geq \sqrt{n(2m_S + |S|) - \alpha(n)(|\lambda_1| - |\lambda_n|)^2}$, where $\alpha(n) = n[\frac{n}{2}](1 - \frac{1}{n}[\frac{n}{2}])$.*

Proof. Taking $a_i = |\lambda_i|$, $b_i = |\lambda_i|$, $a = b = |\lambda_n|$ and $A = B = |\lambda_1|$ in Lemma 4.6, we obtain

$$\left| n\sum_{i=1}^{n} |\lambda_i|^2 - \left(\sum_{i=1}^{n} \lambda_i\right)^2 \right| \leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2 \tag{3}$$

but

$$\sum_{i=1}^{n} |\lambda_i|^2 = 2m_S + |S|.$$

Inequality (3) becomes $n(2m_S + |S|) - [E_p(G \oplus S)]^2 \leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2$,
$E_p(G \oplus S) \geq \sqrt{n(2m_S + |S|) - \alpha(n)(|\lambda_1| - |\lambda_n|)^2}$ where $\alpha(n) = n[\frac{n}{2}](1 - \frac{1}{n}[\frac{n}{2}])$, where $[x]$ denotes the integral part of a real number. □

Theorem 4.8. *Let $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n| > 0$ be a non-increasing order of eigenvalues of $G \oplus S$. Then*

$$E_p(G \oplus S) \geq \frac{n(2m_S + |S| + |\lambda_1||\lambda_n|)}{|\lambda_1| + |\lambda_n|}.$$

Proof. Let $a_i \neq 0$, b_i, r and R be real numbers satisfying $ra_i \leq b_i \leq Ra_i$, then the following inequality holds [Theorem 2,[15]].

$$\sum_{i=1}^{n} b_i^2 + rR\sum_{i=1}^{n} a_i \leq (r+R)\sum_{i=1}^{n} a_i b_i$$

By putting $b_i = |\lambda_i|$, $a_i = 1, r = |\lambda_n|$ and $R = |\lambda_1|$,

$$\sum_{i=1}^{n} |\lambda_i|^2 + |\lambda_1||\lambda_n| \sum_{i=1}^{n} 1 \leq (|\lambda_1| + |\lambda_n|) \sum_{i=1}^{n} |\lambda_i|$$

$$\sum_{i=1}^{n} |\lambda_i|^2 + |\lambda_1||\lambda_n| n \leq (|\lambda_1| + |\lambda_n|) E_p(G \oplus S)$$

$$E_p(G \oplus S) \geq \frac{n(2m_S + |S| + |\lambda_1||\lambda_n|)}{|\lambda_1| + |\lambda_n|}.$$

□

5. Energy of partial complements of some families of Graphs

Theorem 5.1. *Let $K_{1,n-1} \oplus S$ be the partial complement of star graph with $|S| = k$ vertices including central vertex. Then $E_p(K_{1,n-1} \oplus S) = (k-1) + \sqrt{1 + 4(n-k)}$.*

Proof. $A_p(K_{1,n-1} \oplus S) = \begin{bmatrix} \mathbf{1} & O_{1\times(k-1)} & \mathbf{1}_{1\times(n-k)} \\ O_{(k-1)\times 1} & J_{k-1} & O_{(k-1)\times(n-k)} \\ \mathbf{1}_{(n-k)\times 1} & O_{(n-k)\times(k-1)} & O_{n-k} \end{bmatrix}_{n\times n}$ is the partial complement matrix of $K_{1,n-1} \oplus S$. The result is proved by showing $AW = \lambda W$ for certain vector W and by making use of fact that the geometric multiplicity and algebraic multiplicity of each eigenvalue λ is same, as A_p is real and symmetric.

Let $W = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$ be an eigenvector of order n partitioned conformally with A_p.

Consider

$$[A_p(K_{1,n-1} \oplus S) - \lambda I] \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{bmatrix} (1-\lambda)X + \mathbf{1}_{1\times(n-k)} Z \\ (J_{k-1} - \lambda I)Y \\ \mathbf{1}_{(n-k)\times 1} X - \lambda I_{n-k} Z \end{bmatrix}_{n\times n} \tag{4}$$

Case 1: Let $X = 0$, $Y = \mathbf{1}_{k-1}$ and $Z = O_{n-k}$.

From equation (4), we have $(1-\lambda)0 + \mathbf{1}_{1\times(n-k)} O_{n-k} = 0$,

$(J_{k-1} - \lambda I)\mathbf{1}_{k-1} = (k-1)\mathbf{1}_{k-1} - \lambda \mathbf{1}_{k-1} = (k-1-\lambda)\mathbf{1}_{k-1}$.

This implies that, $\lambda = k-1$ is an eigenvalue with multiplicity atleast one.

and $\mathbf{1}_{(n-k)\times 1} 0 - \lambda I_{n-k} O_{n-k} = 0$

Case 2: Let $X = \lambda$, where λ is any root of the equation

$$\lambda^2 - \lambda - (n-k) = 0. \tag{5}$$

Let $Y = O_{k-1}$ and $Z = \mathbf{1}_{n-k}$.

From equation (4), $(1-\lambda)\lambda + \mathbf{1}_{1\times(n-k)} \mathbf{1}_{n-k} = \lambda - \lambda^2 + (n-k)$.

From equation (5), $\lambda = \dfrac{1 + \sqrt{1 + 4(n-k)}}{2}$ and $\lambda = \dfrac{1 - \sqrt{1 + 4(n-k)}}{2}$ are the eigenvalues both with multiplicty of atleast one.

Case 3: Let $X = 0$, $Y = Y_j$ and $Z = O_{n-k}$

where Y_j denote the vector with 1^{st} element 1, j^{th} element -1, where $j = 2, 3, \ldots, k-1$ and remaining $0's$.

From equation (4), $(1-\lambda)0 + \mathbf{1}_{1\times(n-k)} O_{n-k} = 0$

$(J_{k-1} - \lambda I)Y_j = O_{k-1} - \lambda Y_j = -\lambda Y_j$.
Hence, $\lambda = 0$ is an eigenvalue with multiplicity of atleast $k-2$, as there are $(k-2)$ independent eigenvectors of the form Y_j.
$\mathbf{1}_{(n-k)\times 1}0 - \lambda I_{n-k}O_{n-k} = 0$.
Case 4: Let $X = 0$, $Y = O_{k-1}$ and $Z = Z_j, j = 2,3,\dots,n-k$ denote the vector with 1^{st} element 1, j^{th} element -1 and remaining $0's$.
From equation (4), $(1-\lambda)0 + \mathbf{1}_{1\times(n-k)}Z_j = 0$, $(J_{k-1} - \lambda I)O_{k-1} = 0$ and $\mathbf{1}_{(n-k)\times 1}0 - \lambda I_{n-k}Z_j = -\lambda I_{n-k}Z_j = -\lambda Z_j$.
So $\lambda = 0$ is an eigenvalue with multiplicity of atleast $n-k-1$, as there are $(n-k-1)$ independent eigenvectors of the form Z_j.
Since the order of star graph is n, partial complement spectrum is

$$\begin{pmatrix} 0 & k-1 & \frac{1+\sqrt{1+4(k-n)}}{2} & \frac{1-\sqrt{1+4(k-n)}}{2} \\ n-3 & 1 & 1 & 1 \end{pmatrix}$$

and partial complement energy is $E_p(K_{1,n-1} \oplus S) = (k-1) + \sqrt{1+4(n-k)}$. □

Theorem 5.2. *Partial complement energy of complete graph K_n with $|S| = k$ vertices is $E_p(K_n \oplus S) = (n-2) + \sqrt{(n-k)^2 + 4(k-1)n - 4(k^2-k-1)}$.*

Proof. $A_p(K_n \oplus S) = \begin{bmatrix} I_{k\times k} & J_{k\times(n-k)} \\ J_{(n-k)\times k} & (J-I)_{(n-k)\times(n-k)} \end{bmatrix}_{n\times n}$ is the partial complement matrix of $(K_n \oplus S)$. The result is proved by showing $AZ = \lambda Z$ for certain vector Z and by making use of fact that the geometric multiplicity and algebraic multiplicity of each eigenvalue λ is same, as A_p is real and symmetric.
Let $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $2n$ partitioned conformally with A_p.
Consider

$$(A_p(K_n \oplus S) - \lambda I)\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{bmatrix} [(1-\lambda)I]X + JY \\ JX + [J-(\lambda+1)I]Y \end{bmatrix} \tag{6}$$

Case 1: Let $X = X_j = e_1 - e_j, j = 2,3,\dots,k$ and $Y = O_{n-k}$.
From equation (6), $[1-\lambda I]X_j + JO_{n-k} = (1-\lambda)X_j$ then, $\lambda = 1$ is the eigenvalue with mutiplicity of atleast $(k-1)$ since there are $(k-1)$ independent vectors of the form X_j.
Case 2: Let $X = O_{k-1}$ and $Y = Y_j = e_1 - e_j, j = 2,3,\dots,n-k$.
From equation (6), $[J-(\lambda+1)I]Y_j = -(\lambda+1)Y_j$.
so $\lambda = -1$ is the eigenvalue with mutiplicity of atleast $(n-k-1)$ since there are $n-k-1$ independent vectors of the form Y_j.
Case 3: Let $X = \mathbf{1}_k$ and $Y = \left(\frac{k+\lambda-1}{\lambda+1}\right)\mathbf{1}_{n-k}$, where λ is any root of the equation

$$\lambda^2 + \lambda(k-n) + (k^2-k-1) + n(1-k) = 0 \tag{7}$$

From equation (6),

$$[(1-\lambda)I_k]\mathbf{1}_k + J_{k\times(n-k)}\left(\frac{k+\lambda-1}{\lambda+1}\right)\mathbf{1}_{n-k} = (1-\lambda)\mathbf{1}_k + \left(\frac{k+\lambda-1}{\lambda+1}\right)(n-k)\mathbf{1}_k$$
$$= \left\{1-\lambda+\left(\frac{k+\lambda-1}{\lambda+1}\right)(n-k)\right\}\mathbf{1}_k$$
$$= \frac{\lambda^2+\lambda(k-n)+(k^2-k-1)+n(1-k)}{\lambda+1}\mathbf{1}_k$$

So

$$\lambda = \frac{n-k}{2} + \frac{\sqrt{(n-k)^2+4((k-1)n-(k^2-k-1))}}{2}$$

and

$$\lambda = \frac{n-k}{2} - \frac{\sqrt{(n-k)^2+4((k-1)n-(k^2-k-1))}}{2}$$

are the eigenvalues with multiplicity of atleast one.
Thus partial complement spectrum of complete graph is

$$\left\{\begin{matrix} 1 & k-1 \\ -1 & n-k-1 \\ \frac{n-k}{2} + \frac{\sqrt{(n-k)^2+4((k-1)n-(k^2-k-1))}}{2} & 1 \\ \frac{n-k}{2} - \frac{\sqrt{(n-k)^2+4((k-1)n-(k^2-k-1))}}{2} & 1 \end{matrix}\right\}. \qquad \square$$

Theorem 5.3. *Let $K_{l,m} \oplus S$ be partial complement of complete bipartite graph with partites V_1 and V_2 of l and m vertices respectively. Let $\langle S\rangle$ be an induced subset of V which consists of p vertices of V_1 and $k-p$ vertices of V_2. Then characteristic polynomial of $K_{l,m} \oplus S$ is $\lambda^{n-3}[\lambda^3 - k\lambda^2 - \{l(n-l)+2p(p-k)\}\lambda + \{(lk-p^2)n - lk(k+l-2p)\}]$.*
(i) *When $l = p$, $E_p(K_{l,m} \oplus S) = (k-l) + \sqrt{l^2+4l(n-k)}$.*
(ii) *When $l = k$, $E_p(K_{l,m} \oplus S) = \sqrt{k^2+4k(n-k)}$.*

Proof. The characteristic polynomial of $K_{l,m} \oplus S$ is given by
$|A_p(K_{l,m} \oplus S) - \lambda I| =$

$$\left|\begin{array}{c|c|c|c} (J-\lambda I)_{p\times p} & O_{p\times l-p} & O_{p\times k-p} & J_{p\times M} \\ \hline O_{l-p\times p} & (O-\lambda I)_{l-p\times l-p} & J_{l-p\times k-p} & J_{l-p\times M} \\ \hline O_{k-p\times p} & J_{k-p\times l-p} & (J-\lambda I)_{k-p\times k-p} & O_{k-p\times M} \\ \hline J_{M\times p} & J_{M\times l-p} & O_{m\times k-p} & (O-\lambda I)_{M\times M} \end{array}\right|_{n\times n}$$

where $M = (n-l)-(k-p)$.
Step 1: Applying row operation $R_i \longrightarrow R_i - R_{i+1}$, for $i = 1,2,\ldots,p-1,p+1,\ldots,l-p-1,l-p+1,\ldots,k-p-1,k-p+1,\ldots,M-1$ for the above determinant, we get $\lambda^{n-4}\det(B)$.
Step 2: In $\det(B)$, performing $C_i \longrightarrow C_i + C_{i+1} + C_{i+2} + \ldots + C_n$ for $i = 1,2,\ldots,n$, we get $\det(C)$.
Step 3: Observe that each row of $\det(C)$ has only one non zero integer 1 except the row namely $p, l-p, k-p$ and M. On expansion of $\det(C)$, it reduces to order 4. By simplifying we get

$$(8) \qquad |A_p(K_{l,m} \oplus S) - \lambda I| = \lambda^{n-3}[\lambda^3 - k\lambda^2 - \{l(n-l)+2p(p-k)\}\lambda + \{(lk-p^2)n - lk(k+l-2p)\}].$$

Then by putting $l = p$ and $l = k$ in equation (8), we get $\lambda^{n-3}[\lambda - (k - l)][\lambda^2 - l\lambda - l(n-k)]$ and $\lambda^{n-3}[\lambda^2 - k\lambda - k(n-k)]$ respectively.
Hence the results (i) and (ii) follows □

Theorem 5.4. *Let $K_{2\times n} \oplus S$ be the partial complement of a crown graph with $|S| = k$.*
(i) *$E_p(K_{2\times n} \oplus S) = 2(n-1) + \sqrt{n^2 + 4(n-1)^2}$ for $k = n$.*
(ii) *$E_p(K_{2\times n} \oplus S) = 2(2n-1)$ for $k = 2n$.*

Proof. **(i)** $A_p = \begin{bmatrix} J_n & (J-I)_n \\ (J-I)_n & O_n \end{bmatrix}_{2n\times 2n}$ is the partial complement matrix of $K_{2\times n} \oplus S$. The result is proved by showing $AZ = \lambda Z$ for certain vector Z and by making use of fact that the geometric multiplicity and algebraic multiplicity of each eigenvalue λ is same, as A_p is real and symmetric. Let $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $2n$ partitioned conformally with A_p.
Consider

$$(A_p - \lambda I)\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{bmatrix} [(J-\lambda)I]X + (J-I)Y \\ (J-I)X - \lambda I Y \end{bmatrix} \tag{9}$$

Case 1: Let $X = \mathbf{1}_n$ and $Y = \dfrac{\lambda - n}{n-1}\mathbf{1}_n$, where λ is any root of the equation

$$\lambda^2 - n\lambda - (n-1)^2 = 0. \tag{10}$$

From equation (9),

$$(J - \lambda I)\mathbf{1}_n + (J-I)\left(\frac{\lambda - n}{n-1}\right)\mathbf{1}_n = (n-\lambda)\mathbf{1}_n + (\lambda - n)\mathbf{1}_n = 0$$

and

$$\begin{aligned}(J-I)\mathbf{1}_n - \lambda I\left(\frac{\lambda-n}{n-1}\right)\mathbf{1}_n &= (n-1)\mathbf{1}_n - \lambda\left(\frac{\lambda-n}{n-1}\right)\mathbf{1}_n \\ &= \frac{(n-1)^2 - \lambda(\lambda-n)}{n-1}\mathbf{1}_n \\ &= \frac{\lambda^2 - n\lambda - (n-1)^2}{n-1}\mathbf{1}_n.\end{aligned}$$

Hence , $\lambda = \frac{n}{2} + \frac{\sqrt{n^2+4(n-1)^2}}{2}$ and $\lambda = \frac{n}{2} - \frac{\sqrt{n^2+4(n-1)^2}}{2}$ are eigenvalues both with multiplicity of atleast one.
Case 2: Let $X = X_j = e_1 - e_j, j = 2, 3, \ldots, n$ and $Y = -\lambda X_j$, where λ is any root of

$$\lambda^2 - 1 = 0. \tag{11}$$

From equation (9), $(J - \lambda I)X_j + (J-I)(-\lambda X_j) = -\lambda X_j + \lambda X_j = 0$ and $(J-I)X_j + \lambda^2 I_n X_j = (1-\lambda^2)X_j$.
Thus, $\lambda = 1$ and $\lambda = -1$ are the eigenvalues both with multiplicity of atleast $(n-1)$ as there are $(n-1)$ independent eigenvectors of the form X_j. Thus partial complement spectrum of crown graph with $|S| = n$ is

$\begin{pmatrix} 1 & -1 & \frac{n}{2}+\frac{\sqrt{n^2+4(n-1)^2}}{2} & \frac{n}{2}-\frac{\sqrt{n^2+4(n-1)^2}}{2} \\ n-1 & n-1 & 1 & 1 \end{pmatrix}$

and its partial complement energy is $E_p(K_{2\times n}\oplus S) = 2(n-1)+\sqrt{n^2+4(n-1)^2}$.

(ii) Partial complement adjacency matrix of $K_{2\times n}\oplus S$ with $|S| = 2n$ is given by

$A_p(K_{2\times n} \oplus S) = \begin{bmatrix} J_{n\times n} & I_{n\times n} \\ I_{n\times n} & J_{n\times n} \end{bmatrix}_{2n\times 2n}$

where J is the matrix of all $1's$ and I is the identity matrix.

Let

$$[A_p(K_{2\times n} \oplus S) - \lambda I] \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} (J-\lambda I)X - IY \\ -IX + (J-\lambda I)Y \end{bmatrix} \tag{12}$$

Case 1: Consider $X = X_j = e_1 - e_j, j = 1, 2, \ldots, n$ and $Y = \lambda X_j$, where λ is any root of

$$\lambda^2 - 1 = 0. \tag{13}$$

From equation (12),

$$(J-\lambda I)X_j + IX_j = \lambda X_j + \lambda X_j = 0$$

and also $IX_j + (J-\lambda I)X_j = X_j - \lambda^2 X_j = (1-\lambda^2)X_j$.

So $\lambda = 1$ and $\lambda = -1$ are the eigenvalues both with multiplicity of atleast $n-1$ as there are $n-1$ indepedent eigenvectors X_j.

Case 2: $X = \mathbf{1}_n$ and $Y = (\lambda - n)\mathbf{1}_n$, where λ is any root of the equation

$$\lambda^2 - 2n\lambda + n^2 - 1 = 0. \tag{14}$$

From equation (12),

$$(J-\lambda I)\mathbf{1}_n + I(\lambda - n)\mathbf{1}_n = (n - \lambda + \lambda - n)\mathbf{1}_n = 0$$

and

$$I\mathbf{1}_n + (J-\lambda I)(\lambda - n)\mathbf{1}_n = [1-(\lambda-n)^2]\mathbf{1}_n = [\lambda^2 - 2n\lambda + n^2 - 1]\mathbf{1}_n.$$

Hence, $\lambda = n+1$ and $\lambda = n-1$ are the eigenvalues both with multiplicity of atleast one.

So partial complement spectrum of crown graph with $|S| = 2n$ is

$\begin{pmatrix} 1 & -1 & n-1 & n+1 \\ n-1 & n-1 & 1 & 1 \end{pmatrix}$

and its partial complement energy is $E_p(K_{2\times n} \oplus S) = 2(2n-1)$. □

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